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Characterizing Global Optimality for DC Optimization Problems under Convex Inequality Constraints*

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Abstract. Characterizations of global optimality are given for general difference convex (DC) optimization problems involving convex inequality constraints. These results are obtained in terms of ε-subdifferentials of the objective and constraint functions and do not require any regularity condition. An extension of Farkas' lemma is obtained for inequality systems involving convex functions and is used to establish necessary and sufficient optimality conditions. As applications, optimality conditions are also given for weakly convex programming problems, convex maximization problems and for fractional programming problems.

Key words: DC optimization, generalized Farkas' lemma, convex maximization, convex analysis.

1. Introduction

Consider the following general constrained difference convex (DC) global optimization problem

$$(P) \quad \text{global minimize } p(x) - f(x)$$
$$\text{subject to } g_i(x) \leqslant 0, i \in I,$$

where I is an arbitrary index set and the functions $p, f, g_i : \mathbb{R}^n \to \mathbb{R}, i \in I$, are continuous convex functions. The optimization model problem (P) covers, in particular, the standard convex programming problem and convex maximization problems. The model problem appears in various practical applications (see [12, 13, 19]) and in the design of numerical algorithms of several general global optimization problems. Many difficult combinatorial problems, such as nonlinear integer programming problems [20] and quadratic assignment problems [19], and various nonconvex minimization problems such as nonlinear fractional programming problems can be reformulated and solved as constrained difference convex minimization problems (P).

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A great deal of attention has recently been focused on the techniques of solving constrained global optimization problems, based on cutting plane methods, outer approximation techniques and branch and bound methods, see [12, 19]. However, the study of dual conditions characterizing global optimality that is vital for any non-convex optimization problem has so far been limited mainly to convex maximization problems with linear constraints [8], sublinear constraints [15, 16] or difference sublinear constraints [4, 16]. More recently, global optimality conditions for fractional programming problems with finitely many convex constraints using a generalized Slater regularity condition have been given in Ellala [3]. These dual conditions are expressed utilizing ε-subdifferentials. From the theoretical and computational view point, a detailed examination and the development of necessary and sufficient conditions for global optimization problems is of great importance, see [1, 19, 25]. These conditions can also be used as stopping rules in numerical procedures, such as branch and bound methods, for solving global optimization problems.

In this paper, we establish dual conditions characterizing global optimality of the model problem (P). The conditions are given in terms of ε-subdifferentials and they do not require any regularity hypothesis. An extension of Farkas' lemma for inequality systems involving convex functions allows us to obtain the necessary and sufficient dual optimality conditions for the convex inequality constrained problem (P). As applications, we also present optimality conditions for problems with weakly convex ([24]) objective functions, for convex maximization problems and for a class of fractional programming problems. The technical tools used in our approach are ε-subdifferentials and conjugate functions.

The outline of the paper is as follows. In the next section, we develop a version of Farkas' lemma for systems involving convex functions, and in Section 3 we establish necessary and sufficient optimality conditions for a general global difference convex minimization problem with convex constraints. In the appendix, we provide a proof of the extended Farkas' lemma used in Section 3 and related details on solvability of convex inequality systems.

2. ε-Subdifferentials and Farkas' Lemma

We begin this section by presenting definitions of the Fenchel–Moreau conjugate and the ε-subdifferential and their relationships. Throughout this paper X shall denote a real Banach space. The continuous dual space to X will be denoted by X' and will be endowed with the weak* topology. For a set $D \subset X$ we shall denote the *closure* and *convex hull* of D by $cl\, D$ and $co\, D$ respectively. The *cone generated* by the set D is denoted cone $D := \cup_{\alpha \geqslant 0} \alpha D$. The *closed convex cone generated* by D is denoted by $cl(cocone\, D)$.

Let $f : X \to R$ be a continuous convex function. Then the *conjugate function* of f, $f^* : X' \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(\xi) = \sup_{x \in X} \{\xi(x) - f(x)\}.$$

The *epigraph* of f, epi f, is defined by

$$\text{epi } f = \left\{ \begin{pmatrix} x \\ r \end{pmatrix} \in X \times \mathbb{R} | f(x) \leqslant r \right\}.$$

If $h(x) = f(x) - \alpha, \alpha \in \mathbb{R}$ then it is easy to see that epi $h^* =$ epi $f^* + \{\binom{0}{\alpha}\}$. For $\varepsilon \geqslant 0$, the ε-subdifferential of f at $a \in X$, $\partial_\varepsilon f(a)$, is given by

$$\partial_\varepsilon f(a) = \{v \in X' | (\forall x \in X) f(x) - f(a) \geqslant v(x-a) - \varepsilon\}.$$

Then, ε-subdifferential is a non-empty convex weak* closed subset of X'. Moreover,

$$\bigcap_{\varepsilon > 0} \partial_\varepsilon f(a) = \partial f(a),$$

the latter set denoting the usual convex subdifferential of f at a. For a detailed discussion of ε-subdifferentials and conjugate functions, see Hiriart-Urruty and Lemarechal [10], see also [21, 25].

If f is sublinear (i.e. convex and positively homogeneous of degree one) then $\partial_\varepsilon f(0) = \partial f(0)$, for every $\varepsilon \geqslant 0$, and epi $f^* = \partial f(0) \times \mathbb{R}_+$. For a closed subset C of X, and $\varepsilon \geqslant 0$, the *ε-normal set* of C at $a \in X$, denoted by $N_\varepsilon(a, C)$, is given by

$$N_\varepsilon(a, C) = \{v \in X' | (\forall x \in C) v(x-a) \leqslant \varepsilon\}.$$

Note that ε-normal set of C at a is the ε-subdifferential of the indicator function $\delta(x|C)$ at a, where

$$\delta(x|C) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{otherwise.} \end{cases}$$

The following lemma shows how the ε-subdifferential of f is related to epi f^*.

LEMMA 2.1. *If $f : X \to \mathbb{R}$ is a continuous convex function and if $a \in X$ then*

$$\text{epi } f^* = \bigcup_{\varepsilon \geqslant 0} \left\{ \begin{pmatrix} v \\ \varepsilon + v(a) - f(a) \end{pmatrix} \middle| v \in \partial_\varepsilon f(a) \right\}$$

Proof. Let $\binom{u}{r} \in \operatorname{epi} f^*$. Then, $f^*(u) \leqslant r$. From the definition of conjugate function, for each $x \in X$,

$$f^*(u) \geqslant u(x) - f(x);$$

thus, for each $x \in X, u(x) - f(x) \leqslant r$. Let $\varepsilon_0 = r + f(a) - u(a) \geqslant 0$. So, $r = \varepsilon_0 - f(a) + u(a)$. Now, for each $x \in X$,

$$f(x) - f(a) \geqslant u(x) - r - f(a) = u(x - a) - \varepsilon_0;$$

thus, $u \in \partial_{\varepsilon_0} f(a)$. Hence,

$$\operatorname{epi} f^* \subset K := \bigcup_{\varepsilon \geqslant 0} \left\{ \begin{pmatrix} v \\ \varepsilon + v(a) - f(a) \end{pmatrix} \middle| v \in \partial_\varepsilon f(a) \right\}.$$

Conversely, let $\binom{u}{r} \in K$. Then, there exists $\varepsilon_0 \geqslant 0$ such that $u \in \partial_{\varepsilon_0} f(a)$ and $r = -f(a) + u(a) + \varepsilon_0$. Therefore,

$$f^*(u) + f(a) - u(a) \leqslant \varepsilon_0,$$

and so,

$$f^*(u) \leqslant \varepsilon_0 + u(a) - f(a) = r;$$

and the proof is completed. □

From Lemma 2.1, we see that if $f : X \to \mathbb{R}$ is a continuous convex function then

$$\operatorname{epi} f^* = \bigcup_{\varepsilon \geqslant 0} \partial_\varepsilon f(0) \times \{\varepsilon - f(0)\}.$$

If, in particular, f is a continuous sublinear function then

$$\operatorname{epi} f^* = \partial f(0) \times \mathbb{R}_+,$$

where $\mathbb{R}_+$ is the set of non-negative real numbers. If $h(x) = f(x) - k, k \in \mathbb{R}$ then $\operatorname{epi} h^* = \partial f(0) \times [k, \infty)$.

We now state a version of Farkas' lemma that provides a dual characterization of solvability for convex inequality systems. Such a form of Farkas' lemma can be deduced from the results of Ha [6] and Gwinner [5]. Moreover, an elegant framework for dual characterizations of solvability of general infinite nonlinear inequality systems, including convex systems, has been recently given in Rubinov, Glover and Jeyakumar [22]. However, for completeness we provide a new direct proof of this result in the Appendix along with a consistency result for convex inequality systems.

THEOREM 2.1. *Let I be an arbitrary index set; let $f, g_i : X \to \mathbb{R}, i \in I$ be continuous convex functions. Suppose that the system*

$$i \in I, \quad g_i(x) \leqslant 0,$$

is consistent. Then

$$\forall i \in I, \quad g_i(x) \leqslant 0 \Longrightarrow f(x) \leqslant 0.$$

if and only if

$$\operatorname{epi} f^* \subset cl \left(\operatorname{cocone} \bigcup_{i \in I} \operatorname{epi} g_i^* \right).$$

Proof. See the appendix for details. □

COROLLARY 2.1. *Let I be an arbitrary index set, let for each $i \in I, f, g_i : X \to \mathbb{R}$ be continuous convex functions and let $\gamma \in \mathbb{R}$. Suppose that the system*

$$i \in I, \quad g_i(x) \leqslant 0$$

is consistent. Then, the following statements are equivalent

$$\forall i \in I, \quad g_i(x) \leqslant 0 \Longrightarrow f(x) \leqslant \gamma \tag{1}$$

$$\forall \delta \geqslant 0, \quad \partial_\delta f(0) \times \{\delta + \gamma - f(0)\} \subset cl \left(\operatorname{cocone} \bigcup_{i \in I} \operatorname{epi} g_i^* \right). \tag{2}$$

Proof. Let $h(x) = f(x) - \gamma$. Then, $h(x)$ is a continuous convex function and the statement (1) is equivalent, by Theorem 2.1, to

$$\operatorname{epi} h^* \subseteq cl \left(\operatorname{cocone} \bigcup_{i \in I} \operatorname{epi} g_i^* \right).$$

Now by Lemma 2.1,

$$\operatorname{epi} h^* = \bigcup_{\delta \geqslant 0} \left\{ \begin{pmatrix} u \\ \delta - f(0) \end{pmatrix} \middle| (u \in \partial_\delta f(0) \right\} + \begin{pmatrix} 0 \\ \gamma \end{pmatrix}.$$

Thus (1) is equivalent to

$$\bigcup_{\delta \geqslant 0} \partial_\delta f(0) \times \{\delta + \gamma - f(0)\} \subseteq cl \left(\operatorname{cocone} \bigcup_{i \in I} \operatorname{epi} g_i^* \right). \qquad \square$$

COROLLARY 2.2. *Let I be an arbitrary index set; let, for each $i \in I, f, g_i \to \mathbb{R}$ be continuous convex functions. Suppose that the system*

$$i \in I, \quad g_i(x) \leqslant 0$$

is consistent. Then, the following statements are equivalent

$$\forall i \in I, \quad g_i(x) \leqslant 0 \Longrightarrow f(x) \leqslant f(0) \tag{3}$$

$$(\forall \delta \geqslant 0) \quad (\forall u \in \partial_\delta f(0)) \quad (u, \delta) \in cl\left(\text{cocone} \bigcup_{i \in I} \text{epi}\, g_i^*\right). \tag{4}$$

Proof. The conclusion follows easily from Theorem 2.1 by taking $\gamma = f(0)$ and noting that the statement (2) now reduces to

$$\bigcup_{\delta \geqslant 0} \partial_\delta f(0) \times \{\delta\} \subset cl\left(\text{cocone} \bigcup_{i \in I} \text{epi}\, g_i^*\right). \qquad \square$$

It is worth observing that if f is sublinear then we get

$$\forall i \in I, \quad g_i(x) \leqslant 0 \Longrightarrow f(x) \leqslant 0$$

if and only if

$$\partial f(0) \times \mathbb{R}_+ \subset cl\left(\text{cocone} \bigcup_{i \in I} \text{epi}\, g_i^*\right).$$

Note that Corollary 2.2 extends also the classical infinite dimensional Farkas' lemma given for cones. To see this let $S \subset Y$ be a closed convex cone with *dual cone* $S^* = \{v \in X' : (\forall x \in S) v(x) \geqslant 0\}$; let $A : X \to Y$ be a continuous linear mapping. Then, $Ax \in -S$ if and only if, for each $\lambda \in S^*$, $\lambda Ax \leqslant 0$. Hence, for $v \in X'$, we have $[Ax \in -S \Longrightarrow v(x) \leqslant 0]$ if and only if

$$\{v\} \times \mathbb{R}_+ \subset cl\left(\text{cocone} \bigcup_{\lambda \in S^*} \{A^T\lambda\} \times \mathbb{R}_+\right) = cl A^T(S^*) \times \mathbb{R}_+;$$

We wish also to point out that versions of Farkas' lemma for sublinear, i.e. convex and positively homogeneous functions, have recently been obtained completely in terms of subdifferentials (see [4]). However, it can be shown that similar versions completely in terms of subdifferentials do not extend to convex functions. We have established in this section extensions using ε-subdifferentials.

3. DC Minimization with Convex Constraints

In this section, we are concerned with applications of the generalized Farkas' lemma to obtain a complete characterization of optimality for global optimization problems with convex constraints.

Consider again the problem

$$(P) \quad \text{global minimize } p(x) - f(x)$$
$$\text{subject to } g_i(x) \leqslant 0, i \in I,$$

where X is a Banach space, I is an arbitrary index set and $p, f, g_i : X \to \mathbb{R}, i \in I$, are continuous convex functions. Many difficult global optimization problems can be reformulated and solved as difference convex minimization problems. Most of the constrained optimization problems studied in the literature of global optimization deal with finitely many linear constraints. Here we examine a general model problem (P) with explicit convex inequality constraints. Moreover, the number of constraints is not restricted to be finite.

Before presenting optimality conditions that characterize the global optimum of our model problem (P), let us look at how the ε-normal set of the set, described by a system of infinite convex inequalities, can be characterized in terms of ε-subdifferentials. This will play a crucial role in the development of optimality conditions for constrained optimization problems. In [10] such a characterization is given for the set which is described by finitely many linear inequalities.

THEOREM 3.1. *Let $g_i : X \to \mathbb{R}, i \in I$, be continuous convex functions. Let*

$$C = \{x \in X | (\forall i \in I) g_i(x) \leqslant 0\};$$

let $a \in C$ and $\varepsilon \geqslant 0$. Then $v \in N_\varepsilon(a, C)$ if and only if

$$\begin{pmatrix} v \\ v(a) + \varepsilon \end{pmatrix} \in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right).$$

Proof. Note that $v \in N_\varepsilon(a, C)$ if and only if

$$\forall i \in I \quad g_i(x) \leqslant 0 \Longrightarrow v(x) \leqslant v(a) + \varepsilon.$$

From Theorem 2.1, we get

$$\begin{pmatrix} v \\ v(a) + \varepsilon \end{pmatrix} \in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right). \quad \square$$

It is worthwhile noting that the above characterization of the ε-normal set in terms of approximate subdifferentials did not require any regularity condition. In Hiriart-Urruty and Lemarechal (see [10, p. 127] and Ellala [3]), the generalized Slater condition was used to establish such a characterization for problems involving *finitely* many constraints. This approach uses the characterization of approximate optimality, i.e. ε-optimality of convex programming problems (see also [25]).

THEOREM 3.2. *Let $p, f, g_i : X \to \mathbb{R}, i \in I$, be continuous convex functions and let a be a feasible point of (P). Then, a is a global minimum for (P) if and only if*

$$(\forall \varepsilon \geqslant 0)(\forall v \in \partial_\varepsilon f(a))(\exists 0 \leqslant \gamma \leqslant \varepsilon, u \in \partial_\gamma p(a)$$

$$\begin{pmatrix} v-u \\ v(a)-u(a)+\varepsilon-\gamma \end{pmatrix}$$
$$\in cl\left(\text{cocone}\bigcup_{\substack{i\in I\\ \delta\geqslant 0}}\left\{\begin{pmatrix} u_i \\ \delta+u_i(a)-g_i(a)\end{pmatrix}\middle| u_i\in\partial_\delta g_i(a)\right\}\right).$$

Proof. The point a is a global minimum of (P) if and only if a is a global minimum of the unconstrained problem

global minimize $p(x)+\delta(x|C)-f(x)$
subject to $x\in X$.

Now from a result of Hiriart-Urruty [8, 9] (see also [23]), a is a global minimum of the preceding unconstrained problem if and only if for each $\varepsilon\geqslant 0$.

$$\partial_\varepsilon f(a)\subset\partial_\varepsilon(p+\delta(\cdot|C))(a).$$

Since the function $p(x)+\delta(x|C)$ is finite at a and f is continuous,

$$\partial_\varepsilon f(a)\subset\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon,\\ \varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0}}\{\partial_{\varepsilon_1}p(a)+N_{\varepsilon_2}(a,C)\}.$$

The conclusion then follows (with $\gamma=\varepsilon_2$) from Theorem 3.1 by noting that $v\in N_\varepsilon(a,C)$ if and only if

$$\begin{pmatrix} v \\ v(a)+\varepsilon \end{pmatrix}\in cl\left(\text{cocone}\bigcup_{\substack{i\in I\\ \delta\geqslant 0}}\left\{\begin{pmatrix} u_i \\ \delta+u_i(a)-g_i(a)\end{pmatrix}\middle| u_i\in\partial_\delta g_i(a)\right\}\right).\ \square$$

As immediate applications of Theorem 3.1 we include the following:

3.1. Weakly Convex Programming

We now see how global optimality of a weakly convex program [24] can be characterized. Let us first recall that a continuous function $h(x)$ is weakly convex if it can be written as $h(x)=p(x)-\frac{\rho}{2}\|x\|^2$ for some continuous convex function p and $\rho\geqslant 0$. Consider the following weakly convex program

(WC) global minimize $p(x)-\dfrac{\rho}{2}\|x\|^2$
subject to $g_i(x)\leqslant 0, i\in I$,

where $\rho\geqslant 0$.

THEOREM 3.3. *Let $p, g_i : X \to \mathbb{R}, i \in I$, be continuous convex functions and let a be a feasible point of (WC). Then, a is a global minimum for (WC) if and only if for each $\varepsilon \geqslant 0$ and for each $v \in X'$ with $v = \rho w$ and $\sqrt{\rho}\|w - a\| \leqslant \sqrt{2\varepsilon}$ there exist $0 \leqslant \gamma \leqslant \varepsilon$ and $u \in \partial_\gamma p(a)$ satisfying*

$$\begin{pmatrix} v - u \\ v(a) - u(a) + \varepsilon - \gamma \end{pmatrix} \in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right).$$

Proof. The proof follows from the previous result by taking $f(x)$ as $\frac{\rho}{2}\|x\|^2$ and by noting that

$$\partial_\varepsilon f(a) = \{\rho w | \sqrt{\rho}\|w - a\| \leqslant \sqrt{2\varepsilon}\}. \qquad \square$$

3.2. Convex Programming

We shall briefly consider the special case of (P) in which $f \equiv 0$. In this case, the problem (P) becomes the standard convex programming problem with infinitely many constraints [17, 18].

$$\begin{aligned} \text{(CP)} \quad & \text{minimize } p(x) \\ & \text{subject to } g_i(x) \leqslant 0, i \in I. \end{aligned}$$

For this problem, Theorem 3.3 with $\rho = 0$ provides a characterization of optimality without constraint qualification for the convex programming problem (CP). The optimality condition becomes the following:

$$(\forall \varepsilon \geqslant 0)(\exists 0 \leqslant \gamma \leqslant \varepsilon, u \in \partial_\gamma p(a))$$

$$\begin{pmatrix} -u \\ -u(a) + \varepsilon - \gamma \end{pmatrix} \in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right).$$

It is not difficult to show that this condition is equivalent to the existence of $u \in \partial p(a)$ satisfying

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix} \in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right).$$

3.3. Convex Global Maximization

We now focus on the convex maximization problem

$$\text{(MP)} \quad \text{maximize } f(x)$$
$$\text{subject to } g_i(x) \leqslant 0, i \in I,$$

where $f, g_i : X \to \mathbb{R}, i \in I$, are continuous convex functions. The following result illustrates the relationship between the value of the problem (MP) and the dual optimality conditions.

THEOREM 3.4. *Let $f, g_i, i \in I$ be continuous convex functions, let a be a feasible point and let $\beta \in \mathbb{R}$. Then*

$$\sup\{f(x) | g_i(x) \leqslant 0, \quad i \in I\} \leqslant \beta \tag{5}$$

if and only if for each $\varepsilon \geqslant 0$ and for each $v \in \partial_\varepsilon f(a)$

$$\begin{pmatrix} v \\ v(a) + \varepsilon + \beta - f(a) \end{pmatrix}$$
$$\in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right). \tag{6}$$

Proof. The statement (5) is equivalent to the implication that

$$\forall i \in I, g_i(x) \leqslant 0 \Longrightarrow f(x) \leqslant \beta. \tag{7}$$

The conclusion follows by applying the solvability result, Theorem 2.1, replacing $f(x)$ by $f(x) - \beta$ and by using Lemma 2.1. □

The characterization of the global maximum of (MP) is easily obtained from the above result as follows.

COROLLARY 3.1. *For the problem (MP), assume that f and for each $i \in I, g_i$, are continuous convex functions and that $a \in X$ is a feasible point. Then, a is a global maximum of (MP) if and only if for each $\varepsilon \geqslant 0$ and $v \in \partial_\varepsilon f(a)$,*

$$\begin{pmatrix} v \\ v(a) + \varepsilon \end{pmatrix} \in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right).$$

Proof. Since the problem (MP) attains its maximum at a,

$$\sup\{f(x)|g_i(x) \leqslant 0, i \in I\} = f(a) = \beta.$$

Hence, the optimality conditions follow from (6) by substituting $f(a)$ for β. □

Note that Corollary 3.1 can also be derived as a special case of Theorem 3.2 or as an application of Theorem 3.2 in [7]. We provide a simple example (see [15]) to illustrate the nature of the conditions in Corollary 3.1.

EXAMPLE 3.1. Consider the following simple problem.

$$\text{Maximize } f(x) \text{ subject to } 0 \leqslant x \leqslant 1,$$

where $f : \mathbb{R} \to \mathbb{R}$ is the polyhedral convex function defined by $f(x) = 0$, for $x \leqslant 0, x$, for $x \in (0,1)$ and $2x - 1$ for $x \geqslant 1$. Clearly the global maximum occurs at $x = 1$ and $\partial_\varepsilon f(1) = [\max\{1-\varepsilon, 0\}, 2]$. Note that $\partial_\varepsilon f(1) = \partial_{\bar{\varepsilon}} f(1)$ for all $\varepsilon \geqslant \bar{\varepsilon} = 1$ (see [8]). The conclusion of Corollary 3.1 is easily shown to be satisfied since for each $\varepsilon \geqslant 0$ and $v \in \partial_\varepsilon f(1)$,

$$(v, v+\varepsilon) \in \{(x,y) \in \mathbb{R}^2 : x \leqslant y\}.$$

3.4. Fractional Programming

Consider now the following constrained fractional programming problem, studied in Ellala [3]

$$\text{(FP)} \quad \text{global minimize } \frac{p(x)}{f(x)}$$
$$\text{subject to } g_i(x) \leqslant 0, \quad i \in I,$$

where $p, f, g_i : X \to \mathbb{R}, i \in I$ are continuous convex functions with $f(x) > 0$ and $p(x) \geqslant 0$ on the feasible set.

THEOREM 3.5. *For the problem (FP), assume that a is a feasible point. Then, a is a global minimum for (FP) if and only if*

$$(\forall \varepsilon \geqslant 0)(\forall v \in \partial_\varepsilon(p(a)f)(a))(\exists 0 \leqslant \gamma \leqslant \varepsilon, w \in \partial_{\gamma/f(a)} p(a))$$

$$\begin{pmatrix} v - f(a)w \\ v(a) - f(a)w(a) + \varepsilon - \gamma \end{pmatrix}$$
$$\in cl \left(\text{cocone} \bigcup_{\substack{i \in I \\ \delta \geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\} \right).$$

Proof. Since $f(x) > 0$ and $p(x) \geqslant 0$ on the feasible set of (FP), the point a is a global minimum of (FP) if and only if it is a global minimum of the following difference convex minimization problem

$$\begin{aligned}&\text{global minimize } f(a)p(x) - p(a)f(x)\\&\text{subject to } g_i(x) \leqslant 0, \quad i \in I,\end{aligned}$$

The conclusion then follows from Theorem 3.2 by noting that $f(a) > 0$ and

$$\partial_\gamma(f(a)p)(a) = f(a)\partial_{\gamma/f(a)}p(a). \qquad \square$$

Note that if $p(a) > 0$ then the above optimality condition can be simplified as follows:

$$(\forall\varepsilon \geqslant 0)(\forall v \in \partial_{\varepsilon/p(a)}f(a)(\exists 0 \leqslant \gamma \leqslant \varepsilon, w \in \partial_{\gamma/f(a)}p(a))$$

$$\begin{pmatrix} p(a)v - f(a)w \\ p(a)v(a) - f(a)w(a) + \varepsilon - \gamma \end{pmatrix}$$
$$\in cl\left(\text{cocone} \bigcup_{\substack{i\in I\\ \delta\geqslant 0}} \left\{ \begin{pmatrix} u_i \\ \delta + u_i(a) - g_i(a) \end{pmatrix} \middle| u_i \in \partial_\delta g_i(a) \right\}\right).$$

4. Conclusions and Further Research

In this paper, we have shown how global optimality of certain difficult nonconvex optimization problems with convex inequality constraints can be characterized using ε-subdifferentials. Our results do not assume any regularity condition (or constraint qualification) and provide asymptotic necessary and sufficient optimality conditions. Such dual conditions completely characterizing optimality for infinite convex programming problems using ε-subdifferentials were given recently in [14]. A constraint qualification is generally assumed to obtain necessary optimality conditions for convex programming problems, see [18, 10]. We have shown that by presenting the conditions in asymptotic form and by using ε-subdifferentials the standard constraint qualification can be dropped, this is true even for certain non-convex problems involving convex inequality constraints. It would be interesting to know under what conditions these asymptotic optimality conditions collapse to a Lagrangian type condition or to a non-asymptotic form.

We developed these optimality results by first studying solvability characterizations of convex inequality systems, such as generalized Farkas' lemma, and then applied these results to appropriate global optimization problems. This approach to studying constrained global optimization suggests interesting research questions.

For instance, in many practical optimization problems not only is the objective function difference convex but also certain constraints are also of this type such as the convex optimization problems with a reverse convex constraint studied in [13]. We can provide dual descriptions of global optimality for such problems if characterizations of solvability of appropriate inequality systems are known. Hence, the following questions arise. Can we establish complete dual characterizations for the solvability of the following inequality system?

$$\forall i \in I, g_i(x) \leqslant h_i(x) \Longrightarrow f(x) \leqslant 0,$$

where f, g_i and h_i are continuous convex functions. If so, do such characterizations have non-asymptotic forms under regularity conditions? These questions are worth investigating as the answers would allow applications to a wide class of nonconvex global optimization problems.

5. Appendix – Solvability of Convex Inequality Systems

In this section we will provide a proof of Theorem 2.1. To facilitate the argument we shall break the proof into a number of preliminary lemmas. It should be noted that these preliminary results are interesting in their own right as we shall demonstrate with a simple application of Lemma 4.1 to characterizing consistency of convex inequality systems.

LEMMA 5.1. *Let $f : X \to R \cup \{+\infty\}$ be l.s.c. and convex. Then the following statements are equivalent:*

(i) $\exists x \in X, f(x) \leqslant 0$,

(ii) $(0, -1) \notin cl\ cone\ epi\ f^*$.

Proof. Suppose that (ii) is valid, then by the separation theorem there is a $(x, \alpha) \in X \times \mathbb{R}$ such that

$$-\alpha < 0, \quad (\forall (u, \gamma) \in A) u(x) + \gamma\alpha \geqslant 0$$

where $A = cl(cone\, \text{epi}\, f^*)$. Let $\bar{x} = x/\alpha$, so that

$$(\forall (u, \gamma) \in A) u(\bar{x}) + \gamma \geqslant 0.$$

Thus, for any $u \in \text{dom}\, f^*$,

$$u(\bar{x}) + f^*(u) \geqslant 0$$
$$\Longrightarrow u(-\bar{x}) - f^*(u) \leqslant 0.$$

Hence, $f(-\bar{x}) = \sup_u[u(-\bar{x}) - f^*(u)] \leqslant 0$, and so (i) is valid. Thus (ii) implies (i).

Suppose that (ii) is not valid. Then there are nets (β_i) and (ε_i) in $\mathbb{R}$ with $\beta_i > 0, (u_i) \subset \text{dom}\, f^*$ and, for all $i, (u_i, \varepsilon_i) \in \text{epi}\, f^*$ such that $\beta_i u_i \to 0$ and $\beta_i \varepsilon_i \to -1$. Thus, for convenience, we can assume $\varepsilon_i < 0$ for all i. It follows that $\varepsilon_i \geqslant f^*(u_i)$ for all i.

Take any $x \in \operatorname{dom} f$, then, for any i:

$$\begin{aligned} f(x) &= \sup_u [u(x) - f^*(u)] \\ &\geqslant u_i(x) - f^*(u_i) \\ &\geqslant u_i(x) - \varepsilon_i. \end{aligned}$$

In particular, it follows that,

$$\beta_i f(x) \geqslant \beta_i u_i(x) - \beta_i \varepsilon_i.$$

However, $\beta_i u_i(x) - \beta_i \varepsilon_i \to 1$, thus it follows that $f(x) > 0$ and so (i) cannot be satisfied. □

LEMMA 5.2. *Let $f : X \to R \cup \{+\infty\}$ be l.s.c. and convex and let and $D = \{x \in X | f(x) \leqslant 0\}$ be nonempty. Then,*

$$(-D \times \{1\})^* = cl(cone\, \operatorname{epi} f^*).$$

Proof. Let $(u, \alpha) \in \operatorname{epi} f^*$ then $f^*(u) \leqslant \alpha$. Let $x \in D$, then

$$\begin{aligned} u(x) - \alpha &\leqslant u(x) - f^*(u) \\ &\leqslant \sup_u \{u(x) - f^*(u)\} \\ &= f(x) \\ &\leqslant 0. \end{aligned}$$

Thus, for each $x \in D, (u, \alpha)(-x, 1) \geqslant 0$. Hence $(u, \alpha) \in (-D \times \{1\})^*$ and so $\operatorname{epi} f^* \subseteq (-D \times \{1\})^*$. Thus

$$cl(cone\, \operatorname{epi} f^*) \subseteq (-D \times \{1\})^*.$$

To establish the reverse inclusion, we use the separation theorem. Suppose that $(u, \alpha) \notin A$ where $A = cl(cone\, \operatorname{epi} f^*)$. Since D is nonempty we have, by Lemma 5.1, that $(0, -1) \notin A$. Then the entire line segment connecting the points (u, α) and $(0, -1)$ is not in A. We now apply the separation theorem [11] to the compact convex set consisting of the line segment joining these points and the closed convex cone A. Thus there is a point (x, β) such that, for all $\delta \in [0, 1]$:

$$[\delta(u, \alpha) + (1 - \delta)(0, -1)](x, \beta) < 0, \tag{8}$$

and

$$v(x) + \gamma\beta \geqslant 0, \quad \forall (v, \gamma) \in A. \tag{9}$$

By letting $\delta = 0$ in (8) it follows that $\beta > 0$ and letting $\delta = 1$ gives $u(x/\beta) + \alpha < 0$. Let $x' = x/\beta$ then we have $u(-x') > \alpha$. Now for any $v \in \operatorname{dom} f^*$, $(v, f^*(v)) \in A$ in (9) and we find that

$$\begin{aligned} & v(x') + f^*(v) \geqslant 0 \\ \Longrightarrow\ & v(-x') - f^*(v) \leqslant 0 \\ \Longrightarrow\ & \sup_v \{v(-x') - f^*(v)\} \leqslant 0 \\ \Longrightarrow\ & f(-x') \leqslant 0. \end{aligned}$$

Thus $x' \in -D$ and $(u, \alpha) \notin (-D \times \{1\})^*$. Thus the result is established. □

Note that the conclusion of Lemma 5.2 shows also that the following statements are equivalent:
(i) $f(x) \leqslant 0 \Longrightarrow u(x) \leqslant \alpha$.
(ii) $(u, \alpha) \in cl(cone\,\text{epi}\, f^*)$,
where $u \in X'$ and $\alpha \in R$. This is another simple form of Farkas' lemma involving a convex function. Observe that if for each $i \in I, g_i$ is a l.s.c. convex function and if $g = \sup_i g_i$, then

$$\text{epi}\, g^* = cl\left(co \bigcup_{i \in I} \text{epi}\, g_i^*\right).$$

We are now ready to present the proof of Theorem 2.1 using Lemmas 5.1 and 5.2. For convenience, we restate Theorem 2.1 here.

THEOREM 5.1. *Let f and, for each $i \in I, g_i$ be l.s.c. convex functions. Assume that $\{x \in X | g_i(x) \leqslant 0, \forall i \in I\}$ is nonempty. Then the following are equivalent:*
(i) $\forall i \in I, g_i(x) \leqslant 0 \Longrightarrow f(x) \leqslant 0$
(ii) $\text{epi}\, f^* \subseteq cl\,(\text{cocone} \bigcup_{i \in I} epi\ g_i^*)$.
Proof. Let $g = \sup_i g_i$ and $D = \{x \in X | g(x) \leqslant 0\}$, then (i) is equivalent to:

$$\begin{aligned} & [g(x) \leqslant 0 \Longrightarrow f(x) \leqslant 0] \\ \Longrightarrow\ & [g(x) \leqslant 0 \Longrightarrow u(x) \leqslant \alpha] \quad \forall (u, \alpha) \in \text{epi}\, f^* \\ \Longrightarrow\ & (\forall (u, \alpha) \in \text{epi}\, f^*)(u, \alpha) \in (-D \times \{1\})^* && (10) \\ \Longrightarrow\ & \text{epi}\, f^* \subseteq (-D \times \{1\})^* && (11) \end{aligned}$$

Note that (10) follows by the definition of a dual cone and (11) is equivalent to (ii) follows since D is nonempty (by assumption) using Lemma 5.2. □

We can deduce the following consistency result for convex inequality systems using Lemma 5.1.

THEOREM 5.2. *For each $i \in I$ let g_i be l.s.c. and convex. Then exactly one of the following statements holds.*
(i) $(\exists x \in X)(\forall i \in I) g_i(x) \leqslant 0$
(ii) $(0, -1) \in cl\,(\text{cocone} \bigcup_{i \in I} \text{epi}\, g_i^*)$.

Proof. Let $g = \sup_i g_i$. Then, the conclusion follows by applying Lemma 5.1 and by noting that

$$\operatorname{epi} g^* = cl \left(co \bigcup_{i \in I} \operatorname{epi} g_i^* \right). \qquad \square$$

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